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THE CASE FOR DUALITY VIOLATIONS IN THE ANALYSIS OF HADRONIC τ DECAYS

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We discuss why, in the determination of $\alpha_s(m_\tau^2)$ from hadronic τ decays, two important assumptions made in most of previous analyses, namely the neglect of higher-dimension condensates and of Duality Violations (DVs), have introduced uncontrolled systematic errors into this determination. Although the use of pinched weights is usually offered as a justification of these assumptions, we explain why it is not possible to simultaneously suppress these two contributions; particularly since the Operator Product Expansion is expected to be an asymptotic, rather than a convergent expansion. There is not only experimental and theoretical evidence for DVs, but they also affect the extraction of $\alpha_s(m_\tau^2)$.

Keywords: QCD; τ decays; α_s .

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1. Introduction

In 1992, Braaten, Narison and Pich,¹ and later on Le Diberder and Pich,² culminated a series of previous investigations^{3–10} on the hadronic decay of the τ lepton

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with an analysis which revealed that this decay is a very interesting one for studying QCD. This led to, in particular, a very competitive determination of α_s ; a determination which has survived until now. However, today, 24 years later, the increase in precision of the experimental data¹¹ and the improved theoretical knowledge¹² has called for a thorough reinvestigation of this type of analysis, which will be discussed in more detail in section 5, and to which we will refer as “the old-strategy analysis.” This reinvestigation of the old-strategy analysis has exposed as two of the main sources of systematic errors the neglect of the contribution from higher-order condensates and the neglect of the contributions from the so-called quark-hadron Duality Violations (DVs).¹² As we will discuss below, these two contributions turn out to be related.

DVs refer to the failure of the Operator Product Expansion (OPE) to properly describe the physics on the Minkowski axis and, in particular, to reproduce the spectral functions which are the primary experimental data. Our modification of the old strategy, consisting of a series of papers dealing with several aspects of the analysis,^{13–17} has culminated in the work of Ref. 18 which has produced, among other things, a new value for $\alpha_s(m_\tau^2)$. This is where we currently stand. A detailed account of this new analysis will be presented in these proceedings by D. Boito.¹⁹ In this article, on the other hand, we will focus on the connection between the contribution from higher-dimension condensates and DVs, and their subsequent impact on the determination of $\alpha_s(m_\tau^2)$.

The $V_\mu = \bar{u}\gamma_\mu d$ and $A_\mu = \bar{u}\gamma_\mu\gamma_5 d$ spectral functions with u, d quark content, $\rho_{V/A;ud}(s)$, can be experimentally determined from the ratio

$$R_{V/A;ud} = \frac{\Gamma[\tau \rightarrow (\text{hadrons})_{V/A;ud}\nu_\tau(\gamma)]}{\Gamma[\tau \rightarrow e\bar{\nu}_e\nu_\tau(\gamma)]} , \quad (1)$$

as,²⁰

$$\frac{dR_{V/A;ud}(s)}{ds} = \frac{12\pi^2|V_{ud}|^2 S_{EW}}{m_\tau^2} \left[w_T(s/m_\tau^2)\rho_{V/A;ud}^{(1+0)}(s) - w_L(s/m_\tau^2)\rho_{V/A;ud}^{(0)}(s) \right] , \quad (2)$$

where S_{EW} is a short-distance electroweak correction and $w_T(x) = (1-x)^2(1+2x)$, $w_L(x) = 2x(1-x)^2$ are polynomials with a double zero at $x = 1$. This is referred to in the usual jargon as “doubly pinched.” As we will see, the use of pinching will be important in the discussion that follows. The spectral functions $\rho^{(1,0)}(s)$ are related to the corresponding two-point current correlators $\Pi_{\mu\nu}^{(1,0)}(q)$ as $\rho^{(1,0)}(s) = (1/\pi)\text{Im}\Pi^{(1,0)}(s)$, where the scalar functions $\Pi^{(1,0)}(q^2)$ are defined as

$$\begin{aligned} \Pi_{\mu\nu}(q) &= i \int d^4x e^{iqx} \langle 0 | T \{ J_\mu(x) J_\nu^\dagger(0) \} | 0 \rangle \\ &= (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^{(1)}(q^2) + q_\mu q_\nu \Pi^{(0)}(q^2) \\ &= (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi^{(1+0)}(q^2) + q^2 g_{\mu\nu} \Pi^{(0)}(q^2) . \end{aligned} \quad (3)$$

J_μ stands for the non-strange V or A current, $\bar{u}\gamma_\mu d$ or $\bar{u}\gamma_\mu\gamma_5 d$, while the superscripts (0) and (1) label spin. In what follows we will always take the 0+1 spin combination

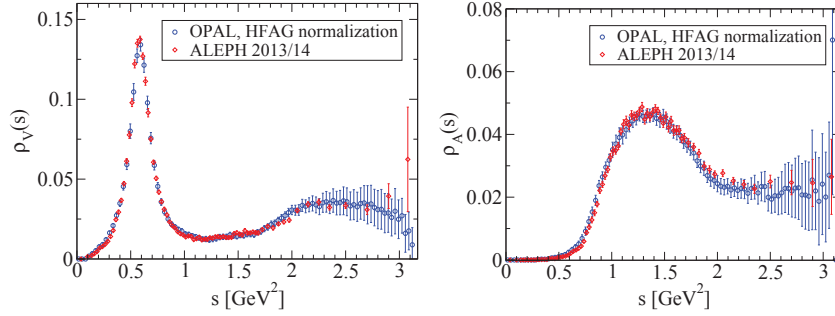


Fig. 1. Opal and Aleph spectral functions $\rho_{V/A}^{(1+0)}(s)$.

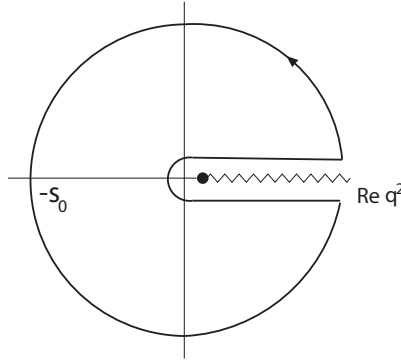


Fig. 2. Contour used in Eq. 4.

because it is free from kinematical singularities. We are using the notation $s = q^2$, so that $q^2 > 0$ in the Minkowski.

The spectral function $\rho_V^{(0)}(s)$ is negligible and $\rho_A^{(0)}(s)$ is dominated by the pion pole, which is known. The combination $\rho_{V/A}^{(1+0)}(s)$ may, therefore, be determined directly from $dR_{V/A;ud}(s)/ds$. This provides the primary experimental data the following analysis will be based on, and it is shown in Fig. 1.^{11,21} As one can see, even above $s \simeq 1.5 \text{ GeV}^2$, the data shows clear oscillations not present in the OPE prediction. The OPE prediction, i.e., perturbation theory plus condensates, in fact, corresponds to a nearly horizontal line with a value of ~ 0.025 , in clear disagreement with the experimental data.

2. Theoretical Foundations

Even though the OPE locally disagrees with the spectral data, one expects that at least globally there might be a better agreement. Therefore, one considers Finite

Energy Sum rules (FESRs) i.e., integrals of the form

$$I_{V/A}^{(w)}(s_0) \equiv \frac{1}{s_0} \int_0^{s_0} ds w(s) \rho_{V/A}(s) = -\frac{1}{2\pi i s_0} \oint_{|s|=s_0} ds w(s) \Pi_{V/A}(s) , \quad (4)$$

valid for any $s_0 > 0$ and any weight $w(s)$ analytic inside and on the contour represented in Fig. 2.^{1,3–10} In the second equality, Cauchy’s theorem has been used on account of the analytic structure of the correlators $\Pi(q^2)$, which have a cut on the positive real axis,^a precisely where the spectral functions are measured. As it stands, Eq. (4) is an exact mathematical statement as long as the exact correlator $\Pi(q^2)$ is used. However, this correlator is not known exactly from QCD. If the scale s_0 is deemed “large enough” (more about the perils of this assumption in the following), it begins to make sense to use of the OPE on the contour and write the right-hand side of Eq. (4) as

$$= -\frac{1}{2\pi i s_0} \oint_{|s|=s_0} ds w(s) \left[\Pi_{V/A}^{OPE}(s) + \Pi_{V/A}^{DV}(s) \right] , \quad (5)$$

where, by definition, $\Pi_{V/A}^{DV}(s) = \Pi_{V/A}(s) - \Pi_{V/A}^{OPE}(s)$. It is precisely the term $\Pi_{V/A}^{DV}(s)$ which will be the subject of the following discussion. The superscript “DV” stands for *Duality Violations* (DVs) and refers to the fact that the correlator computed with the OPE in terms of quark and gluon fields does not equal the hadronic counterpart. Apart from the term encoding DVs, Eqs. (4) and (5) yield the necessary relation to determine QCD parameters (such as α_s , contained in $\Pi_{V/A}^{OPE}$), from the experimental data (contained in $\rho_{V/A}$).

If there were a limit in which $\Pi_{V/A}^{OPE}(s) \rightarrow \Pi_{V/A}(s)$ as, e.g., in a convergent expansion, then one would of course have $\Pi_{V/A}^{DV}(s) \rightarrow 0$. However, there is rather compelling evidence that the OPE is *not* a convergent expansion.^b Perhaps the simplest way to see this is by considering the OPE as an expansion in $1/q^2$, i.e. neglecting all the logarithmic corrections. In this case the OPE becomes a power expansion and, as such, its assumed convergence should happen in a full disc in the complex plane around $1/q^2 = 0$. Clearly, the existence of the cut shown in Fig. 2, running all the way to infinity, contradicts this. The large- N_c limit also suggests the non-convergence of the OPE. In the large- N_c limit the spectral function has poles at arbitrarily high positive q^2 (i.e., the meson masses). Obviously there is no sense in which q^2 is “larger than any meson mass” to make a large- q^2 expansion a convergent one. A somewhat more elaborate argument can also be found in Ref. 22.

If the OPE is not convergent, there is no sense in which s_0 may be “large enough” to guarantee that $\Pi^{DV} \sim 0$ in Eq. (5). In the case of the physical decay of the τ lepton, the scale $s_0 = m_\tau^2$. So, the argument that m_τ is a large scale suppressing the contribution of high-dimension condensates in the OPE, as has been invoked in the literature in the past, is unreliable.

^aThe axial channel also has the pion pole.

^bAlthough, to the best of our knowledge, no formal proof is available.

But not everything is lost. If the OPE is not convergent, the next logical possibility is that it is at least asymptotic. Indeed, asymptotic expansions may have the property of approaching the true function only in an angular sector of the complex plane. The fact that the OPE does a good job describing the correlator in the Euclidean ($q^2 < 0$) while failing in the Minkowski where the cut is located ($q^2 > 0$) would suggest this is the case. This would come at a price, however. Unlike convergent expansions, an asymptotic expansion misses a term relative to the true function. This missing term is of order $e^{-\gamma/\alpha}$ for the case of an expansion parameter $\alpha > 0$. The constant γ depends on the case considered. In the case of renormalons, for instance, one has $\alpha = \alpha_s$ and γ is related to (the inverse of) the beta function. Therefore, in the case of the OPE, since the expansion parameter is $1/q^2$, one should expect a correction on the Minkowski axis like $\Pi^{DV} \sim e^{-\gamma q^2}$, for $q^2 > 0$ and some unknown parameter γ .

This is when “pinching” enters our discussion. If the weight function $w(s)$ in Eq. (5) is chosen with a high-order zero at $s = s_0$, the contribution from the contour in the region around the Minkowski axis, where Π^{DV} has its support, will be suppressed. Pinching, therefore, comes as a potentially useful trick to suppress DVs. Although this may be true qualitatively, at a more quantitative level it is far from clear how much of a suppression one really has for a given order in the pinched polynomial. This is of course crucial when good control over the systematics is needed in a reliable determination of an important parameter of QCD such as α_s .

The above discussion leads to the main theoretical message of this article:

“It is not possible to simultaneously suppress DVs and higher-dimension condensates.”

In other words: pinching comes at a price. Indeed, a high-order zero at $s = s_0$ in the polynomial $w(s)$ is only achieved when $w(s)$ is a high-order polynomial. However, since a term of order s^n selects a condensate of dimension $2(n+1)$ in Eq. (5), a high-order polynomial also selects high-order condensates, which are typically either poorly known or unknown.

If the OPE were a convergent expansion, this would be a minor problem as higher-dimension condensates would contribute less and less. This, in fact, has been the main (explicit or tacit) assumption in most of the literature so far. As the OPE is expected to be an asymptotic expansion, neglecting higher-dimension condensates becomes a very crude approximation and, at the very least, potentially introduces uncontrolled systematics one should avoid in any case.

3. Duality Violations: Generalities

Up to now, there is no theory of DVs in QCD from first principles. So, a quantitative analysis of DVs faces serious theoretical challenges and it is no surprise that DVs have been neglected in the old-strategy analysis of hadronic τ decays. Nevertheless, some progress can be made.

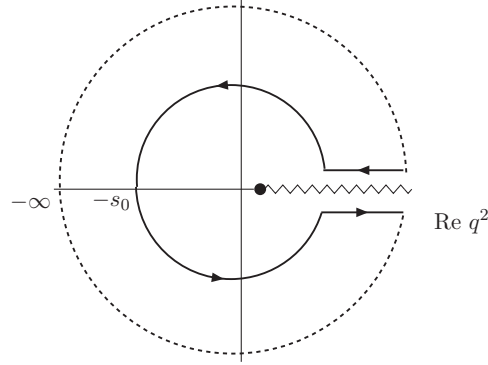


Fig. 3. Contour used to obtain Eq. (6).

First, a generic property of an asymptotic expansion is that it exactly agrees with the true function only for a vanishing value of the expansion parameter. This assures that $\Pi^{DV}(s) \rightarrow 0$ as $s \rightarrow \infty$ (in any direction in the complex plane). Then, using Cauchy's theorem "in reverse" by closing the contour at infinity (which gives a vanishing contribution) as depicted in Fig. 3, one obtains

$$-\frac{1}{2\pi i} \oint_{|z|=s_0} dz w(z) \Pi_{DV}^{V/A}(z) = - \int_{s_0}^{\infty} ds w(s) \frac{1}{\pi} \text{Im} \Pi_{DV}^{V/A}(s). \quad (6)$$

The right-hand side of Eq. 6 makes explicit two important properties of DVs. First, the contribution from DV to FESRs involves an extrapolation from s_0 to infinity, a region which is not covered by experimental τ data. Second, recalling that DVs are exponentially damped, one clearly sees why a zero of the weight $w(s)$ at $s = s_0$, i.e. the lower end of the integration interval, tends to suppress the contribution of the whole integral.^c

4. Duality Violations: An educated guess

Without an explicit representation of DVs it is not possible to go further. We need a concrete parametrization. Keeping in mind the expected qualitative behavior shown in Fig. 4 and the previous discussion, we find it natural to assume¹³

$$\frac{1}{\pi} \text{Im} \Pi_{DV}^{V/A}(s) \simeq e^{-\delta_{V/A}} e^{-\gamma_{V/A} s} \sin(\alpha_{V/A} + \beta_{V/A} s) \quad (7)$$

as the ansatz for the contribution of DVs in Eq. (6), for large enough s . The exponential damping in s stems from the asymptotic property of the OPE, as discussed above. The oscillation is assumed periodic with frequency $\beta_{V/A}$ as inspired by Regge theory. It is clear that the crests seen in Fig. 4 are due to the different resonances,

^cFor a more detailed discussion of this suppression, see section 7.

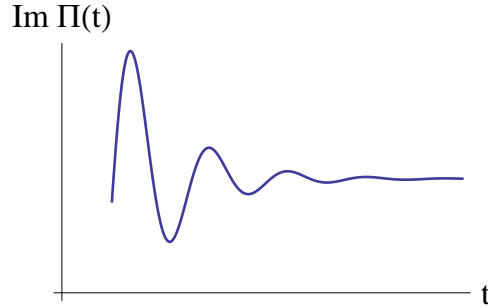


Fig. 4. Schematic representation of the spectrum in Fig. 1, showing DVs.

which would become Dirac deltas in the large- N_c limit. The phenomenological success of Regge theory predicting daughter trajectories i.e., an equal spacing rule for the squares of the resonance masses, suggests a periodic function with a single frequency as a reasonable first approximation. The amplitude of the oscillation is parametrized as $e^{-\delta_{V/A}}$ just for numerical convenience. We emphasize that, although the functional form is the same for V and A , the values of the parameters may differ since the two spectra approach the OPE result differently. This adds $4+4=8$ parameters in total.

Let us conclude this section by noting that a parametrization of the form of Eq. (7) can be explicitly shown to be true in a specific model constructed to study DVs in Ref. 23, and has been applied to determine low-energy constants and condensates in QCD from the $V - A$ correlator by different groups in Refs. 24-28.

5. Critical Review of the “Old-Strategy” Method

The old-strategy method² consists in using five pinched weights $w_{kl}(y) = (1 - y)^2(1 + 2y)(1 - y)^k y^l$, for $(k, l) = \{(0, 0), (1, 0), (1, 1), (1, 2), (1, 3)\}$, with $y = s/s_0$, in the FESRs (4-5) for a *single* value of $s_0 = m_\tau^2$. We emphasize that these polynomials go up to s^7 which, through the FESR (5), requires knowledge of the condensates up to dimension 16. However, only condensates of dimension 4, 6 and 8 are kept. Condensates of higher dimension, 10, 12, 14 and 16, are set to zero by fiat. The argument behind this assumption is that the relevant scale $s_0 = m_\tau^2$ is large enough, and the OPE is convergent enough, to make the contribution of these higher order condensates numerically negligible. However, as we have seen, since the OPE is asymptotic this assumption is dangerous. Furthermore, DVs are also set to zero on the basis of the above weights being doubly and triply pinched. The danger of employing a high degree of pinching (and hence a high degree polynomial) while at the same time neglecting the higher dimension OPE contributions introduced by the use of the higher degree polynomial, is precisely the point of our main theoretical message expressed in Section 2. As we will see, although DVs are indeed suppressed for these

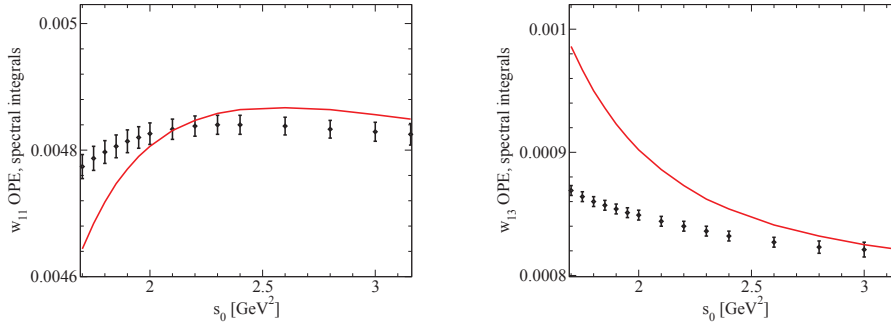


Fig. 5. s_0 dependence of the $V + A$ w_{11} and w_{13} FESR for CIPT from Ref. 11.

weights, they are not totally negligible. In fact, although pinching suppresses DVs, the *amount* of this suppression is not necessarily correlated with the amount of pinching *when a single s_0 is employed*, as in the old-strategy analysis. That is, e.g., a triply pinched weight may have a larger DV contribution than a doubly pinched weight when only a single s_0 value such as $s_0 = m_\tau^2$ is considered (see section 7).

Given the five moments constructed with the five pinched weights w_{kl} defined above, a fit is performed to extract 4 parameters: the condensates of dimension 4, 6 and 8, and $\alpha_s(m_\tau^2)$. This is a fit with only one degree of freedom. The results of this fit¹¹ reveal an inconsistency in the extracted value of the gluon condensate, where in the V channel one has $\langle \frac{\alpha_s}{\pi} GG \rangle = (-0.5 \pm 0.3) \times 10^{-2} \text{ GeV}^4$, ($\chi^2 = 0.43$, $p = 51\%$), while in the $V + A$ channel one has $(-2.0 \pm 0.3) \times 10^{-2} \text{ GeV}^4$, ($\chi^2 = 1.1$, $p = 29\%$), which are incompatible. Clearly the gluon condensate should have come out to be the same. The A channel produces a third incompatible number. Although the V channel result is the one statistically preferred (i.e., a higher p value), it is the $V + A$ channel's result which is chosen for the extraction of QCD parameters and, in particular, of α_s .¹¹

6. Comparing tests

Restricting the analysis to just $s_0 = m_\tau^2$ is dangerous.²⁹ With a fixed value of s_0 there is no way to determine whether neglected, but in fact non-negligible, higher-dimension condensates contaminate the results. The parameters will always do “the best they can” to fit the data at this s_0 . Because the contribution from different condensates scales differently with s_0 , a nontrivial check is then whether these parameters can also describe the data when s_0 is lowered by some amount. If the result of the fit deviates from the data as soon as s_0 is lowered, this is a clear evidence of contamination by unaccounted for effects.

Fig. 5 shows the comparison of the result of the fit in Ref. 11 and the data for two of the weight functions used in their fit: w_{11} and w_{13} . As one can see, the fit representations deviate from the data as soon as s_0 is lowered. For comparison,

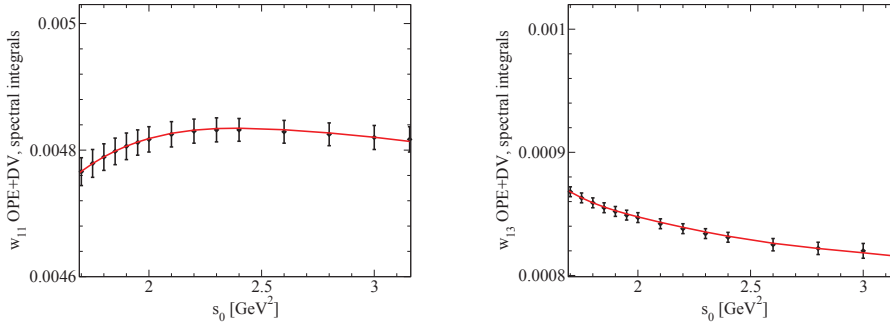


Fig. 6. s_0 dependence of the $V + A$ w_{11} and w_{13} FESR for CIPT from Ref. 18.

Fig. 6 shows the same plots but using the results of the fits obtained in Ref. 18. The difference is striking.

7. Demystifying pinched weights

Fig. 7 shows the result of the integral

$$F^{(w_{kl})}(s_0) = \left| \int_{s_0}^{\infty} \frac{ds}{s_0} w_{kl}\left(\frac{s}{s_0}\right) \frac{1}{\pi} \text{Im}\Pi_{DV}^V(s) \right| \quad (8)$$

as a function of s_0 and the five pinched weights employed in the old-strategy analysis^{2,11} for the values of the DV parameters found in the analysis of Ref. 18. Specifically, the parameters used are $\delta_V \simeq 3.5$, $\gamma_V \simeq 0.62 \text{ GeV}^{-2}$, $\alpha_V \simeq -2.43$ and $\beta_V \simeq 4.32 \text{ GeV}^{-2}$, corresponding to the $s_{\min} = 1.55 \text{ GeV}^2$, FOPT result of Table I in Ref. 18.^d Although w_{00} is only doubly pinched, whereas all the other w_{kl} are triply pinched (see section 5), Fig. 7 shows explicitly how all the triply pinched moments actually produce a *larger* contribution to the FESR from DVs than the doubly pinched weight w_{00} near the value of $s_0 = m_\tau^2 \simeq 3.16 \text{ GeV}^2$ where the old-strategy analysis is made. This is of course against the naive belief that more pinching reduces DVs more, and is a consequence of the DV function in Eq. (7) not having a definite sign. On the other hand, when a window in s_0 is considered, one sees in Fig. 7 how the amplitude of the oscillations for w_{00} is larger than those for the triply pinched weights, as one would naively have guessed. So, again, the lesson is that considering a window in s_0 is safer than a just a single value, even if this is the largest possible, $s_0 = m_\tau^2$.

8. An illustrative exercise: Hunting for systematic errors.

Even though neglecting DVs by fiat is not the best way to quantify their effect on α_s , one might be tempted to think that, if the weight is sufficiently pinched and the

^dSee Ref. 19 for the precise definition of FOPT and CIPT.

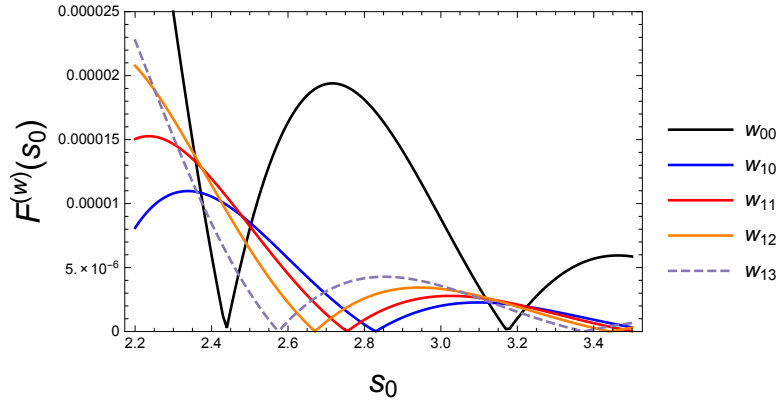


Fig. 7. Contribution from DVs to the FESR for the 5 pinched weights of the old-strategy method, Eq. (8).

FESR shows a good match between the fit and the data as s_0 is lowered from m_τ^2 , the analysis is safe in the sense that systematic effects caused by non-vanishing DVs are numerically negligible, in particular for the $V + A$ channel since the oscillations in the spectrum are smaller there.

Fig. 8 shows the result of a fit to $V + A$, FOPT, for w_{00} (i.e., the same weight as in the decay width) for $1.95 \text{ GeV}^2 \leq s \leq m_\tau^2$, *without* DVs. It is necessary to fit in a window of s_0 because the FESR depends on three parameters: α_s , $C_{6,V+A}$ and $C_{8,V+A}$. The optimal fit, with a high p value of 57%, is obtained using the fit window $2.2 \text{ GeV}^2 < s_0 < m_\tau^2$ and yields

$$\begin{aligned} \alpha_s(m_\tau^2) &= 0.330 \pm 0.006, \\ C_{6,V+A} &= 0.0070 \pm 0.0022 \text{ GeV}^6, \\ C_{8,V+A} &= -0.0088 \pm 0.0042 \text{ GeV}^8. \end{aligned} \quad (9)$$

As one can see, the quality of the fit is excellent and it would make nobody suspect that there is a systematic effect lurking in the results. However, it is there.

To unravel this systematic effect, one may look at the first Weinberg sum rule (WSR) for this case. This is shown in Fig. 9. The value of s_{sw} shown in abscissa is the point up to which the integral in the WSR is taken. Clearly the WSR is violated even at those s_0 values for which there is a good match in Fig. 8. Since the $V-A$ spectral function entering the WSR vanishes to a high level of approximation in the OPE, the non-zero result in Fig. 9 is only due to DVs.

To be able to assess in more quantitative terms the systematic effect on α_s which, after all, is the main objective we are after, one may repeat the fit leading to Fig. 8, but now taking as external input the result of the DV parameters obtained in Ref. 18 from the $s_{\min} = 1.55 \text{ GeV}^2$ fit of Table V, and fitting the OPE parameters, $C_{(6,8),V+A}$ and α_s . The change is dramatic. The new values obtained are now

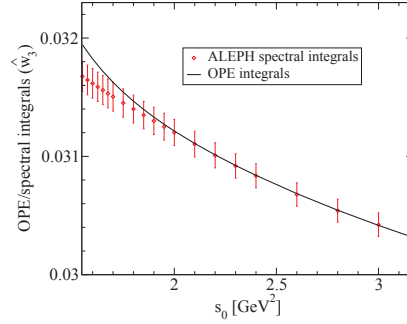


Fig. 8. Comparison of the w_{00} -weighted spectral integral and the OPE integral evaluated with the no-DV fit given in Eq. (9).

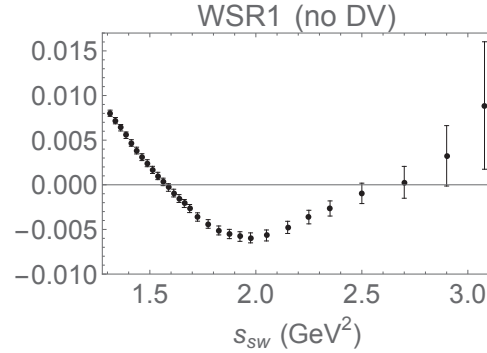


Fig. 9. The 1st Weinberg sum rule for the no-DV fit of Eq. (9).

$$\begin{aligned}
 \alpha_s(m_\tau^2) &= 0.301 \pm 0.006 \pm 0.009, \\
 C_{6,V+A} &= -0.0127 \pm 0.0020 \pm 0.0066 \text{ GeV}^6, \\
 C_{8,V+A} &= 0.0399 \pm 0.0040 \pm 0.021 \text{ GeV}^8,
 \end{aligned} \tag{10}$$

instead of Eq. (9), where the first error is statistical and the second is the one induced by the correlations with the external input DV parameters. As one can see, the condensates even flip sign relative to Eq. (9), and the value of $\alpha_s(m_\tau^2)$ goes *down* by a lot more than the nominal error obtained in the fit (9). Furthermore, unlike the previous case of the WSR in which DVs were neglected (Fig. 9), now the WSR looks like Fig. 10, where s_{sw} is the point at which one switches from the spectral data to the DV parametrization in the infinite integration interval of the WSR. This concludes the exercise.

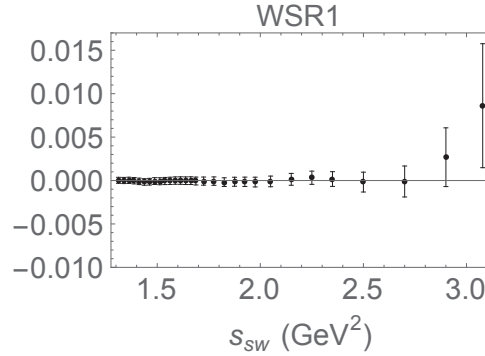


Fig. 10. The 1st Weiberg sum rule from Eq. (10) and the results from Ref. 18.

9. Results

Let us now compare the final results obtained for α_s from τ decays with and without including the DV parametrization (7). With DVs included, we obtain¹⁸

$$\begin{aligned} \alpha_s(m_\tau^2) = 0.296 \pm 0.010 &\longrightarrow \alpha_s(m_Z^2) = 0.1155 \pm 0.0014 \quad (\text{FOPT}) \\ \alpha_s(m_\tau^2) = 0.310 \pm 0.014 &\longrightarrow \alpha_s(m_Z^2) = 0.1174 \pm 0.0019 \quad (\text{CIPT}), \end{aligned} \quad (11)$$

while the old-strategy method produces a shift of the order $+0.03$ higher in α_s at the τ mass and with errors which are about half those shown in Eq. (11).¹¹ The systematic increase in the value of α_s is of course reminiscent of the results found in the exercise of the previous section.

It is sometimes customary to express the result in terms of the R_{V+A} ratio in Eq. (1):

$$R_{V+A} = N_c S_{EW} |V_{ud}|^2 \left(1 + \delta_P + \delta_6 + \delta_8 + \delta_{DV} \right), \quad (12)$$

splitting the QCD contributions into a perturbative term δ_P and a nonperturbative term $\delta_{NP} = \delta_6 + \delta_8 + \delta_{DV}$, where $\delta_{6,8}$ are the contributions from the corresponding dimension in the OPE and δ_{DV} are the DV contributions. In this case we obtain¹⁸

$$\begin{aligned} \delta_{NP} &= 0.020 \pm 0.009 \quad (\text{FOPT}) \\ \delta_{NP} &= 0.016 \pm 0.010 \quad (\text{CIPT}), \end{aligned} \quad (13)$$

to be compared to the value obtained in the old-strategy method, $\delta_{NP} = -0.0064 \pm 0.0013$ (CIPT),¹¹ again with errors which are claimed to be much smaller than those in Eq. (13). Since the results of the old-strategy method did not consider DVs and neglected higher-dimension condensates, these errors are to be considered an underestimate.

Finally, since perturbative and non-perturbative contributions are strongly correlated (since their sum has to equal the spectral function integral) we would like to warn about studies of the perturbative term δ_P which use a previous result for

δ_{NP} as if they were independent of each other. For any new proposal different from CIPT or FOPT to reorganize perturbation theory, a new δ_{NP} should be obtained from a new fit to the FESRs.

10. Conclusions and Outlook

The spectral functions in Fig. 1 show visible oscillations which are due to DVs i.e., terms which go beyond the OPE. DVs, therefore, are not just a question of principle but they also exist in practice. How much DVs impact a determination of $\alpha_s(m_\tau^2)$ cannot be answered by neglecting DVs a priori. Pinching may help suppress DVs (especially if an interval in s_0 is used in the analysis) but one should remember that it is not possible to simultaneously reduce contributions from DVs and the higher-dimension condensates. So, there is a price to pay. Ignoring DVs and/or the contribution from these higher-dimension condensates introduces an unquantified systematic error that has plagued the old-strategy method.^{2,11}

On the other hand, the use of a concrete physically motivated parametrization of DVs allows one to perform a complete reanalysis from scratch and quantitatively check the reliability of all the assumptions made in the old-strategy method. Doing so has unravelled significant systematic effects, the most important of which is a positive shift in $\alpha_s(m_\tau^2)$ of order +0.03 when the old strategy is employed. Therefore, any error claimed smaller than 0.03 in $\alpha_s(m_\tau^2)$ from any previous analysis of hadronic τ decays neglecting DVs must be considered an underestimate. In order to quantify potential systematic errors a concrete parametrization of DVs for QCD must be assumed.

For our parametrization of DVs in Eq. (7), since DVs cannot yet be derived from QCD from first principles, we have used arguments as general as possible to minimize model dependence. In this sense it is important to realize that the old-strategy method is not free of model dependence. This model dependence can be summarized by the following arbitrary choices made there: $e^{-\delta_{V/A}} = 0$ (in Eq. (7)) and $C_{10,12,14,16} = 0$, where the C_n 's are the corresponding OPE condensates. None of these choices are favored by present data or theoretical knowledge. Unlike the case of the old-strategy method, in our case DVs are determined by the data and the analysis passes all known tests up to date, with better performance than the old-strategy analysis.

At this point it is difficult to make progress without a better theoretical understanding of DVs. In this sense, ideas like resurgence³⁰ or functional analysis methods³¹ may be helpful. It would also help to have more detailed inclusive experimental spectral data, such as the one which could, in principle, be obtained from the BABAR and Belle experiments. We would like to encourage our experimental colleagues to provide us with this important information in the near future.

Note added

After completion of this writeup, a new analysis using the “old strategy” appeared.³² Here we just note that this new analysis fails to answer most of the criticisms of the old strategy raised in Sec. VII of Ref. 18, and which are reviewed here. We also point out that even though Ref. 32 dismisses the use of ansatz (7) in the determination of α_s from $V + A$, the authors advocate the use of precisely this type of ansatz in the extraction of low-energy constants and OPE condensates from $V - A$ in Refs. 24 and 28.

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